Available online at www.sciencedirect.com
ScienceDirect
International Journal of Heat and Mass Transfer 50 (2007) 1723–1736
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Non-similar analytic solution for MHD flow and heat transfer in a third-order fluid over a stretching sheet

M. Sajid [a,c], T. Hayat [a,*], S. Asghar [b]

[a] Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan
[b] COMSATS Institute of Information Technology, H-8, Islamabad 44000, Pakistan
[c] Physics Research Division, PINSTECH, P.O. Nilore, Islamabad 44000, Pakistan

Received 10 August 2006; received in revised form 10 October 2006
Available online 15 December 2006

Abstract

This paper investigates the magnetohydrodynamic (MHD) flow and heat transfer characteristics in the presence of a uniform applied magnetic field. The boundary layer flow of a third-order fluid is induced due to linear stretching of a non-conducting sheet. The heat transfer analysis has been carried out for two heating processes, namely (i) with prescribed surface temperature (PST-case) and (ii) prescribed surface heat flux (PHF-case). The governing non-linear differential equations are solved analytically using homotopy analysis method (HAM). The series solutions are developed and the convergence of these solutions is discussed. Velocity and temperature distributions are shown graphically. The numerical values for the skin friction coefficient and the Nusselt number are entered in tabular form. Emphasis has been given to the variations of the emerging parameters such as third-order parameter, magnetic parameter, Prandtl number and the Eckert number. It is noted that the skin friction coefficient decreases as the magnetic parameter or the third grade parameter increases.

Keywords: Heat transfer; Third-order fluid; Boundary layer flow; HAM solution

1. Introduction

Boundary layer flow over a stretching surface is often encountered in many engineering disciplines. The aerodynamic extrusion of plastic sheets, the cooling of an infinite metallic plate in a cooling bath, the boundary layer along a liquid film in condensation process and a polymer sheet or filament extruded continuously from a dye are few examples of practical applications of a continuous flat surface. Many metallurgical processes involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. By drawing such strips in an electrically conducting fluid, the cooling rate can be controlled and product of desired characteristics can be obtained. Further, flow and heat transfer phenomena over stretching surface has promising applications in a number of technological processes including production of polymer films or thin sheets. More specifically, heat transfer analysis plays a vital role during the handling and processing of non-Newtonian fluids. Such analysis in boundary layer flows of non-Newtonian fluids occurs in the design of thrust bearing and radial diffusers, transpiration cooling, drag reduction and thermal recovery of oil. The flows of non-Newtonian fluids [1–8] are very important due to their industrial and technological applications.

Extensive work in the literature has been performed for the boundary layer flow and heat transfer of a viscous and second grade fluids over a stretching surface. Most recently, Liu [9] discussed the flow of a second grade fluid with heat transfer. Although extensive existing investigations of second-order fluid model exhibit normal stresses but for steady flow it does not describe the property of

* Corresponding author. Tel.: +92 51 2275341.
E-mail address: t_pensy@hotmail.com (T. Hayat).

0017-9310/$ - see front matter © 2006 Published by Elsevier Ltd.
doi:10.1016/j.ijheatmasstransfer.2006.10.011

shear thinning or thickening. Therefore some experiments may be well described by third or fourth-order fluids [10,11]. The third-order approximation of a simple fluid exhibits shear dependent viscosity; for a simple-shearing motion ($\mathbf{v} = (\gamma y, 0, 0)$), where γ is the rate of strain. The relation between the shearing stress and the rate of strain is given by $S_{12} = \mu(1 - T_s^2\gamma^2)\gamma$, where T_s is the shear-relaxation time (its reciprocal is the characteristic rate of strain at which the apparent shear viscosity noticeable decreases), and μ is the lower limiting viscosity. Experiments by Bruce [12] and Joseph [13] have shown that there are materials that exhibit: (1) strong normal stresses but are weakly shear-thinning (class 1); (2) roughly equal normal and shearing effects (class 2); and (3) weak normal stresses but are strongly shear-thinning (class 3). The model in the present study is of class 3 fluids (or third order). To the best of our knowledge, no attention has been given to the MHD boundary layer flow and heat transfer analysis of a third-order fluid over a linear stretching sheet. Even, the hydrodynamic boundary layer flow of a third-order fluid caused by a stretching surface without heat transfer is not discussed so far. Thus, the primary objective of this work is to present the analytical solution of such attempt. This problem is not only important because of its technological significance but also in view of the interesting mathematical features presented by the equations governing the flow.

The paper is organized as follows. We start our formulation in Section 2 by defining the continuity, momentum, constitutive equations and boundary conditions in the Cartesian coordinates. In the Section 2.1, we find the analytic solution for the velocity using HAM [14–31]. The expression for the skin friction coefficient is also given in Section 2.2. The energy equation for the thermodynamic third-order fluid is presented in Section 3. Sections 3.1 and 3.2 respectively deal with the boundary conditions and HAM solutions of the temperature distribution for the prescribed surface temperature case and the prescribed heat flux case. In Section 4, we show the convergence of the solution. In Section 5 the results relevant to the graphs are presented. Section 6 synthesis the concluding remarks.

2. Flow analysis

In this section, we consider the MHD flow of a third-order fluid over a stretching sheet with the plane $y = 0$. By applying two equal and opposite forces along the x-axis, the sheet is being stretched with a speed proportional to the distance from the fixed origin $x = 0$. The fluid occupies the half space $y > 0$ and the motion of the otherwise quiescent fluid is induced due to the non-conducting stretching sheet. A magnetic field $\mathbf{B} = (0, B_0, 0)$ is applied and the induced magnetic field is neglected by taking the magnetic Reynolds number very small. Besides, no electric field is applied and the effect of polarization of the ionized fluid is negligible and thus $\mathbf{E} = 0$. It is also well known that the third grade fluid has Cauchy stress tensor $\mathbf{T}$ of the following form:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{1}$$

where the kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined by

$$\begin{aligned} \mathbf{A}_1 &= \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathsf{T}}, \\ \mathbf{A}_2 &= \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^{\mathsf{T}}\mathbf{A}_1, \end{aligned} \tag{2}$$

in which $\mathbf{V}$ is the fluid velocity and $\mathrm{d}/\mathrm{d}t$ is the material derivative.

For steady plane flow the two dimensional equations are of the following form:

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = &-\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma B_0^2}{\rho}u \\ &+ \frac{\alpha_1}{\rho}\left\{\frac{\partial u}{\partial x}\left(13\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u\left(\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^2\partial x}\right)\right. \\ &+ \left(\frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial x^2\partial y}\right) + 2\frac{\partial v}{\partial x}\left(2\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial y}\right) \\ &\left. + 3\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}\right)\right\} + 2\frac{\alpha_2}{\rho}\left\{4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right. \\ &\left. + \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}\right)\right\} \\ &+ \frac{\beta_1}{\rho}\left[31\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} + 19u\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 21u\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3}\right. \\ &+ u^2\frac{\partial^4 u}{\partial x^4} + 2uv\frac{\partial^4 u}{\partial x^3\partial y} + 19v\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^2\partial y} \\ &+ u^2\frac{\partial^4 u}{\partial x^2\partial y^2} + v^2\frac{\partial^4 u}{\partial x^2\partial y^2} + 17v\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} \\ &+ 3u\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 + u\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x\partial y^2} + 2uv\frac{\partial^4 u}{\partial x\partial y^3} \\ &- 3v\frac{\partial^3 u}{\partial x^3}\frac{\partial u}{\partial y} + 6u\frac{\partial^3 u}{\partial x^2\partial y}\frac{\partial u}{\partial y} + 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y} \\ &+ 7v\frac{\partial^3 u}{\partial x\partial y^2}\frac{\partial u}{\partial y} + 3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2} + 3u\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} \\ &+ 6v\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 u}{\partial y^2} - v\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y^3} + v^2\frac{\partial^4 u}{\partial y^4} - 6v\frac{\partial^3 u}{\partial x^3}\frac{\partial v}{\partial x} \\ &+ 5u\frac{\partial^3 u}{\partial x^2\partial y}\frac{\partial v}{\partial x} + 36\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial x} + 4v\frac{\partial^3 u}{\partial x\partial y^2}\frac{\partial v}{\partial x} \end{aligned}$$

$$
\begin{aligned}
&- 5\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 9\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x} + u\frac{\partial^3 u}{\partial y^3}\frac{\partial v}{\partial x}\\
&- 6\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial v}{\partial x}\right)^2 + 8\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial v}{\partial x}\right)^2 - 6v\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\\
&+ 7u\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x^2} + 8\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 5v\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\\
&+ 6\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + 6u\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 4u\frac{\partial u}{\partial y}\frac{\partial^3 v}{\partial x^3}\\
&+ 6u\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial x^3}\Bigg] + \frac{\beta_2}{\rho}\Bigg[48\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} + 8u\left(\frac{\partial^2 u}{\partial x^2}\right)^2\\
&+ 8u\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + 8v\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^2\,\partial y} + 6v\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}\\
&+ 2u\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 - 2v\frac{\partial^3 u}{\partial x^3}\frac{\partial u}{\partial y} + 2u\frac{\partial^3 u}{\partial x^2\,\partial y}\frac{\partial u}{\partial y}\\
&+ 26\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial u}{\partial y} + 2v\frac{\partial^3 u}{\partial x\,\partial y^2}\frac{\partial u}{\partial y} - 2\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)^2\\
&+ 8\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2} + 2v\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 u}{\partial y^2} + 6\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\\
&- 2v\frac{\partial^3 u}{\partial x^3}\frac{\partial v}{\partial x} + 2u\frac{\partial^3 u}{\partial x^2\,\partial y}\frac{\partial v}{\partial x} + 34\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial v}{\partial x}\\
&+ 2v\frac{\partial^3 u}{\partial x\,\partial y^2}\frac{\partial v}{\partial x} - 6\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 14\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial x}\\
&- 4\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial v}{\partial x}\right)^2 + 8\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial v}{\partial x}\right)^2 - 2v\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\\
&+ 4u\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x^2} + 10\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\\
&+ 10\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + 2u\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2u\frac{\partial u}{\partial y}\frac{\partial^3 v}{\partial x^3}\\
&+ 2u\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial x^3}\Bigg] + \frac{\beta_3}{\rho}\Bigg[40\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}\\
&+ 24\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y} + 24\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}\\
&+ 12\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} + 8\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 8\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\\
&+ 8\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2} + 6\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + 6\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2}\\
&- 4\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} - 2\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} - 2\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}\Bigg],
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = &-\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\\
&+ \frac{\alpha_1}{\rho}\Bigg\{\frac{\partial v}{\partial y}\left(13\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) + u\left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial y^2\,\partial x}\right)\\
&+ v\left(\frac{\partial^3 v}{\partial y^3} + \frac{\partial^3 v}{\partial x^2\,\partial y}\right) + 2\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\,\partial y}\right)\\
&+ 3\frac{\partial v}{\partial x}\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}\right)\Bigg\} + 2\frac{\alpha_2}{\rho}\Bigg\{4\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\\
&+ \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\,\partial y}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\,\partial y}\right)\Bigg\}\\
&+ \frac{\beta_1}{\rho}\Bigg[8\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 v}{\partial x^2} + 6v\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 6v\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3}\\
&+ u^2\frac{\partial^4 v}{\partial x^4} + 2uv\frac{\partial^4 v}{\partial x^3\,\partial y} + 4v\frac{\partial v}{\partial x}\frac{\partial^3 u}{\partial y^3} + u^2\frac{\partial^4 v}{\partial x^2\,\partial y^2}\\
&+ v^2\frac{\partial^4 u}{\partial x^2\,\partial y^2} + 3v\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + 3v\left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2\\
&+ 5v\frac{\partial u}{\partial y}\frac{\partial^3 v}{\partial x\,\partial y^2} + 2uv\frac{\partial^4 v}{\partial x\,\partial y^3} + v\frac{\partial^3 v}{\partial x^3}\frac{\partial u}{\partial y}\\
&+ 4u\frac{\partial^3 v}{\partial x^2\,\partial y}\frac{\partial u}{\partial y} + 6\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\,\partial y}\frac{\partial v}{\partial y} + v^2\frac{\partial^4 v}{\partial y^4}\\
&+ 5u\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 9\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 7u\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial x^2\,\partial y}\\
&+ 7v\frac{\partial^2 v}{\partial x\,\partial y}\frac{\partial^2 u}{\partial y^2} + 6u\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y\,\partial x} + 6v\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial x\,\partial y^2}\\
&+ 6\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + 8\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} - u\frac{\partial v}{\partial y}\frac{\partial^3 v}{\partial x^3}\\
&+ 36\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\,\partial y} + 19u\frac{\partial v}{\partial y}\frac{\partial^3 v}{\partial x\,\partial y^2} + 3\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 v}{\partial x^2}\\
&- 6\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 v}{\partial y^2} - 6u\frac{\partial^2 v}{\partial y^2}\frac{\partial^2 u}{\partial y^2} - 5\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}\\
&+ 17u\frac{\partial^2 v}{\partial x\,\partial y}\frac{\partial^2 v}{\partial y^2} + 31\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 v}{\partial y^2} + 19v\left(\frac{\partial^2 v}{\partial y^2}\right)^2\\
&- 6u\frac{\partial u}{\partial y}\frac{\partial^3 v}{\partial y^3} - 3u\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial y^3} + 21v\frac{\partial v}{\partial y}\frac{\partial^3 u}{\partial y^3}\\
&+ v\frac{\partial^3 v}{\partial x^2\,\partial y}\frac{\partial v}{\partial y}\Bigg] + \frac{\beta_2}{\rho}\Bigg[8\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 v}{\partial x^2} + 2v\left(\frac{\partial^2 u}{\partial y^2}\right)^2\\
&+ 2v\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3} + 2v\frac{\partial v}{\partial x}\frac{\partial^3 u}{\partial y^3} + 2u\frac{\partial^3 v}{\partial x^2\,\partial y}\frac{\partial u}{\partial y}\\
&+ 26\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\,\partial y} + 2u\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 6\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 v}{\partial x^2}\\
&+ 14\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 2u\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial x^2\,\partial y} + 4v\frac{\partial^2 v}{\partial x\,\partial y}\frac{\partial^2 u}{\partial y^2}
\end{aligned}
$$

$$+2u\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y\partial x}+2v\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial x\partial y^2}+2v\frac{\partial u}{\partial y}\frac{\partial^3 v}{\partial x\partial y^2}$$
$$+10\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+10\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+34\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}$$
$$+8u\frac{\partial v}{\partial y}\frac{\partial^3 v}{\partial x\partial y^2}+8\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 v}{\partial x^2}-4\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 v}{\partial y^2}$$
$$-2u\frac{\partial^2 v}{\partial y^2}\frac{\partial^2 u}{\partial y^2}-6\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}+6u\frac{\partial^2 v}{\partial x\partial y}\frac{\partial^2 v}{\partial y^2}$$
$$+48\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 v}{\partial y^2}+8v\left(\frac{\partial^2 v}{\partial y^2}\right)^2-2u\frac{\partial u}{\partial y}\frac{\partial^3 v}{\partial y^3}$$
$$-2u\frac{\partial v}{\partial x}\frac{\partial^3 v}{\partial y^3}+8v\frac{\partial v}{\partial y}\frac{\partial^3 v}{\partial y^3}+2v\left(\frac{\partial^2 v}{\partial x\partial y}\right)$$
$$-2\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 v}{\partial y^2}\Bigg]+\frac{\beta_3}{\rho}\left[40\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 v}{\partial y^2}\right.$$
$$+24\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+24\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}$$
$$+12\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+8\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+8\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}$$
$$+8\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 v}{\partial x^2}+6\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 v}{\partial x^2}+6\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 v}{\partial x^2}$$
$$\left.-4\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}-2\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 v}{\partial y^2}-2\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 v}{\partial y^2}\right]. \tag{4}$$

Under the usual boundary layer arguments that u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial p}{\partial x}$ be O(1) and y, v be O(δ) yields the following equations which govern the MHD boundary layer flow [32]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{5}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_1}{\rho}\left[u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial y^3}\right] +\frac{2\alpha_2}{\rho}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{6(\beta_2+\beta_3)}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u, \tag{6}$$

where u and v are the velocities in the x- and y-direction, respectively, ρ is the fluid density, ν (=μ/ρ), is the kinematic viscosity, σ is the electrical conductivity, α_1, α_2, β_1, β_2 and β_3 are material constants. Note that ν and α_i/ρ $(i=1,2)$ being O(δ^2) and β_i/ρ $(i=1,2,3)$ being O(δ^4) and terms of O(δ) are neglected (where δ being the thickness of the boundary layer).

The relevant boundary conditions are

$$u=Bx,\quad v=0\quad \text{at}\quad y=0,\qquad u\to 0\quad \text{as}\quad y\to\infty. \tag{7}$$

Let us introduce the following dimensionless quantities

$$u=Bxf'(x,\eta),\quad v=-\sqrt{B\nu}\left(f(x,\eta)+x\frac{\partial f}{\partial x}\right),\quad \eta=\sqrt{\frac{B}{\nu}}y,\quad \epsilon_1=\frac{B\alpha_1}{\mu},\quad \epsilon_2=\frac{B\alpha_2}{\mu},\quad \phi=\frac{B^2(\beta_2+\beta_3)}{\mu},\quad \phi_1=\frac{Bx^2}{\nu},\quad M^2=\frac{\sigma B_0^2}{\rho B}, \tag{8}$$

where the prime signifies differentiation with respect to η. The mass conservation equation is automatically satisfied, where as Eq. (6) transforms into the following ordinary differential equation:

$$\left[f'''-f'^2+ff''-M^2f'+xf''\frac{\partial f}{\partial x}-xf'\frac{\partial f'}{\partial x}+\epsilon_1\left(2f'f'''-ff^{iv}+xf'''\frac{\partial f'}{\partial x}-xf^{iv}\frac{\partial f}{\partial x}\right)+(3\epsilon_1+2\epsilon_2)f''^2+6\phi\phi_1f'''f''^2\right]=0. \tag{9}$$

The boundary conditions (7) become

$$f=0,\quad f'=1\quad \text{at}\ \eta=0,\qquad f'\to 0\quad \text{as}\ \eta\to\infty. \tag{10}$$

Eqs. (9) and (10) represent a non-linear differential problem, the analytic solution of which is not so easy. In the next section, we will find the HAM solution of the governing non-linear problem.

2.1. Homotopy analytic solution

In order to solve Eqs. (9) and (10) by HAM we choose the initial approximation

$$f_0(\eta)=1-e^{-\eta} \tag{11}$$

and the auxiliary linear operator

$$\mathscr{L}_1(f)=f'''-f' \tag{12}$$

which has the property

$$\mathscr{L}_1[C_1+C_2e^{\eta}+C_3e^{-\eta}]=0, \tag{13}$$

where C_1, C_2 and C_3 are constants.

The zero-order deformation problem is

$$(1-p)\mathscr{L}_1[\hat{f}(x,\eta,p)-f_0(\eta)]=p\hbar_1\mathscr{N}_1[\hat{f}(x,\eta,p)], \tag{14}$$

$$\hat{f}(x,0,p)=0,\quad \hat{f}'(x,0,p)=1,\quad \hat{f}'(x,\infty,p)=0, \tag{15}$$

where

$$\mathcal{N}_1[\hat{f}(x,\eta,p)] = \frac{\partial^3 \hat{f}(x,\eta,p)}{\partial\eta^3} - M^2\frac{\partial \hat{f}(x,\eta,p)}{\partial\eta} - \left(\frac{\partial \hat{f}(x,\eta,p)}{\partial\eta}\right)^2 + \hat{f}(x,\eta,p)\frac{\partial^2 \hat{f}(x,\eta,p)}{\partial\eta^2} + x\hat{f}''(x,\eta,p)\frac{\partial \hat{f}(x,\eta,p)}{\partial x} - x\hat{f}'(x,\eta,p)\frac{\partial \hat{f}'(x,\eta,p)}{\partial x} + (3\epsilon_1+2\epsilon_2)\left(\frac{\partial^2 \hat{f}(x,\eta,p)}{\partial\eta^2}\right)^2 + \epsilon_1\left(2\frac{\partial \hat{f}(x,\eta,p)}{\partial\eta}\frac{\partial^3 \hat{f}(x,\eta,p)}{\partial\eta^3} - \hat{f}(x,\eta,p)\frac{\partial^4 \hat{f}(x,\eta,p)}{\partial\eta^4} + x\hat{f}'''(x,\eta,p)\frac{\partial \hat{f}'(x,\eta,p)}{\partial x} - x\hat{f}^{iv}(x,\eta,p)\frac{\partial \hat{f}(x,\eta,p)}{\partial x}\right) + 6\phi\phi_1\frac{\partial^3 \hat{f}(x,\eta,p)}{\partial\eta^3}\left(\frac{\partial^2 \hat{f}(x,\eta,p)}{\partial\eta^2}\right)^2 \tag{16}$$

is the non-linear differential operator, $p\in[0,1]$ is the embedding parameter and $\hbar_1$ is auxiliary nonzero parameter. For $p=0$ and $p=1$, we respectively have

$$\hat{f}(x,\eta,0) = f_0(\eta), \quad \hat{f}(x,\eta,1) = f(x,\eta). \tag{17}$$

As p increases from 0 to 1, $\hat{f}(x,\eta,p)$ varies from the initial guess $f_0(\eta)$ to the solution $f(x,\eta)$. By Taylor's theorem and Eq. (17), we have

$$\hat{f}(x,\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(x,\eta)\ p^m, \tag{18}$$

where

$$f_m(x,\eta) = \frac{1}{m!}\frac{\partial^m \hat{f}(x,\eta,p)}{\partial p^m}\bigg|_{p=0}. \tag{19}$$

Clearly, the convergence of the series (18) depends on the auxiliary parameter $\hbar_1$. Assume that $\hbar_1$ is selected such that the series (18) is convergent at $p=1$, then due to Eq. (17) we have

$$f(x,\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(x,\eta). \tag{20}$$

Differentiating m-times the zeroth-order deformation Eq. (14) with respect to p and then dividing by $m!$ and finally setting $p=0$ we get the following mth-order deformation problem

$$\mathcal{L}_1[f_m(x,\eta) - \chi_m f_{m-1}(x,\eta)] = \hbar_1 \mathcal{R}_m^1(x,\eta), \tag{21}$$

$$f_m(x,0) = f'_m(x,0) = f'_m(x,\infty) = 0, \tag{22}$$

where

$$\mathcal{R}_m^1(x,\eta) = f'''_{m-1} - sM^2 f'_{m-1} + \sum_{k=0}^{m-1}\left[f_{m-1-k}f''_k - f'_{m-1-k}f'_k + xf''_{m-1-k}\frac{\partial f_k}{\partial x} - xf'_{m-1-k}\frac{\partial f'_k}{\partial x} + \epsilon_1\left(2f'_{m-1-k}f'''_k - f_{m-1-k}f^{iv}_k + xf'''_{m-1-k}\frac{\partial f'_k}{\partial x} - xf^{iv}_{m-1-k}\frac{\partial f_k}{\partial x}\right) + (3\epsilon_1+2\epsilon_2)f''_{m-1-k}f''_k + 8\phi\phi_1 f''_{m-1-k}\sum_{l=0}^{k}\{f'''_l f''_{k-l}\}\right], \tag{23}$$

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{24}$$

We use the symbolic computation software MATHEMATICA to solve the linear equations (21) and (22) up to first few order of approximation and found that the solution of the problem can be expressed as

$$f_m(x,\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n} a^q_{m,n}\eta^q e^{-n\eta}, \quad m \geqslant 0. \tag{25}$$

Now substituting the expression given in Eq. (25) into Eq. (21) one obtains the following recurrence formulae for the coefficients $a^q_{m,n}$ of $f_m(x,\eta)$ as follows:

for $m \geqslant 1$, $0 \leqslant n \leqslant 2m+1$ and $0 \leqslant q \leqslant 2m+1-n$:

$$a^0_{m,0} = \chi_m\chi_{2m+1}a^0_{m-1,0} - \sum_{q=0}^{2m}\Psi^q_{m,1}\mu^q_{1,1} - \sum_{n=2}^{2m+1}\left[(n-1)\Psi^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2m+1-n}\Psi^q_{m,n}((n-1)\mu^q_{n,0} - \mu^q_{n,1})\right], \tag{26}$$

$$a^k_{m,0} = \chi_m\chi_{2m+1-k}a^k_{m-1,0}, \quad 1 \leqslant k \leqslant 2m+1, \tag{27}$$

$$a^0_{m,1} = \chi_m\chi_{2m}a^0_{m-1,1} + \sum_{q=0}^{2m}\Psi^q_{m,1}\mu^q_{1,1} + \sum_{n=2}^{2m+1}\left\{n\Psi^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2m+1-n}\Psi^q_{m,n}(n\mu^q_{n,0} - \mu^q_{n,1})\right\}, \tag{28}$$

$$a^k_{m,1} = \chi_m\chi_{2m-k}a^k_{m-1,1} + \sum_{q=k-1}^{2m}\Psi^q_{m,1}\mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m+1, \tag{29}$$

$$a^k_{m,n} = \chi_m\chi_{2m+1-n-k}a^k_{m-1,n} + \sum_{q=k}^{2m+1-n}\Psi^q_{m,n}\mu^q_{n,k},$$

$$2 \leqslant n \leqslant 2m+1, \quad 0 \leqslant k \leqslant 2m+1-n, \tag{30}$$

where

$$\mu^q_{1,k} = \sum_{p=0}^{q+1-k}\frac{q!}{k!2^{q+1-k-p}}, \ q \geqslant 0, \ 1 \leqslant k \leqslant 2q+1, \tag{31}$$

$$\mu^q_{n,k} = \sum_{r=0}^{q-k}\sum_{p=0}^{q-k-r}\frac{q!}{k!(n-1)^{q+1-k-r-p}n^{r+1}(n+1)^{p+1}},$$

$$0 \leqslant k \leqslant 2q+1-n, \ q \geqslant 0, \ n \geqslant 2 \tag{32}$$

and the related coefficient $\Psi^q_{m,n}$ is defined by

$$\Psi^q_{m,n} = \hbar_1\Big[\chi_{2m+1-n-q}\Big(d^q_{m-1,n} - M^2 b^q_{m-1,n}\Big) + \chi_{2m+2-n-q}\Big\{\delta^q_{m,n} - \Delta^q_{m,n} + \epsilon_1\Big(2\Lambda^q_{m,n} - \Gamma^q_{m,n}\Big) + (3\epsilon_1 + 2\epsilon_2)\lambda^q_{m,n}\Big\} + 6\phi\phi_1\Omega^q_{m,n}\Big]. \quad (33)$$

Here the coefficients $\delta^q_{m,n}$, $\Delta^q_{m,n}$, $\Lambda^q_{m,n}$, $\Gamma^q_{m,n}$, $\lambda^q_{m,n}$ and $\Omega^q_{m,n}$ where $m \geqslant 1$, $0 \leqslant n \leqslant 2m+1$, $0 \leqslant q \leqslant 2m+1-n$ are defined by

$$\delta^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} c^i_{k,j} a^{q-i}_{m-1-k,n-j}, \quad (34)$$

$$\Delta^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} b^i_{k,j} b^{q-i}_{m-1-k,n-j}, \quad (35)$$

$$\Lambda^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} d^i_{k,j} b^{q-i}_{m-1-k,n-j}, \quad (36)$$

$$\Gamma^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} e^i_{k,j} a^{q-i}_{m-1-k,n-j}, \quad (37)$$

$$\lambda^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} c^i_{k,j} c^{q-i}_{m-1-k,n-j}, \quad (38)$$

$$\Omega^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} \times \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} \times d^i_{l,j} c^{t-i}_{k-l,p-j} c^{q-t}_{m-1-k,n-p}, \quad (39)$$

where the related coefficients $b^k_{m,n}$, $c^k_{m,n}$, $d^k_{m,n}$ and $e^k_{m,n}$ are given by

$$b^k_{m,n} = (k+1)a^{k+1}_{m,n} - na^k_{m,n}, \quad (40)$$
$$c^k_{m,n} = (k+1)b^{k+1}_{m,n} - nb^k_{m,n}, \quad (41)$$
$$d^k_{m,n} = (k+1)c^{k+1}_{m,n} - nc^k_{m,n}, \quad (42)$$
$$e^k_{m,n} = (k+1)d^{k+1}_{m,n} - nd^k_{m,n}. \quad (43)$$

For the detailed procedure of deriving the above relations the reader is referred to [16]. With the above recurrence formulae, we can calculate all coefficients $a^k_{m,n}$ using only the first two

$$a^0_{0,0} = 1, \quad a^0_{0,1} = -1, \quad (44)$$

given by the initial guess approximations for the function $f(\eta)$ in Eq. (11). The corresponding Mth-order approximation of Eqs. (9) and (10) is then given by

$$\sum_{m=0}^{M} f_m(x,\eta) = \sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{2M+1} \mathrm{e}^{-n\eta}\left(\sum_{m=n-1}^{2M} \sum_{k=0}^{2m+1-n} a^k_{m,n}\eta^k\right). \quad (45)$$

We obtain in fact the following explicit, totally analytic solution of the present boundary layer flow

$$f(x,\eta) = \sum_{m=0}^{\infty} f_m(x,\eta) = \lim_{M\to\infty}\left[\sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{2M+1} \mathrm{e}^{-n\eta}\left(\sum_{m=n-1}^{2M} \sum_{k=0}^{2m+1-n} a^k_{m,n}\eta^k\right)\right]. \quad (46)$$

2.2. Skin friction

The shear stress τ_w on the surface of the stretching sheet is

$$\tau_w = \left[\mu\frac{\partial u}{\partial y} + \alpha_1\left(2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + u\frac{\partial^2 u}{\partial x\partial y}\right) + 2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3\right]\Bigg|_{y=0}. \quad (47)$$

The local skin friction coefficient or frictional drag coefficient is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho(Bx)^2} \quad (48)$$

which in terms of dimensionless quantities is

$$C_f = 2Re_x^{-1/2}\left[f'' + \epsilon_1\left\{3f'f'' - ff''' + 2xf''\frac{\partial f'}{\partial x} - xf'''\frac{\partial f}{\partial x}\right\} + 2\phi\phi_1 f''^3\right]\Bigg|_{\eta=0}, \quad (49)$$

where $Re_x = Bx^2/\nu$ is the local Reynolds number based on the length scale x.

3. Heat transfer analysis

The energy equation, corresponding to the boundary layer flow of a thermodynamic third grade fluid is

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \alpha_1\left[u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right] + 2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^4, \quad (50)$$

where T is temperature, c_p is the specific heat and k is the thermal conductivity. The boundary conditions depend on the heating process:

3.1. The prescribed surface temperature (PST case)

Here

$$T = T_w = T_\infty + A\left(\frac{x}{l}\right)^2 \quad \text{at} \quad y = 0,$$
$$T \to T_\infty \quad \text{as} \quad y \to \infty, \tag{51}$$

where A is a constant. Defining

$$\theta(x,\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{52}$$

the problem consisting of Eqs. (50) and (51) becomes

$$\theta'' + Pr\left[\left(f\theta' - 2f'\theta + x\theta'\frac{\partial f}{\partial x} - xf'\frac{\partial \theta}{\partial x}\right) + E\left\{f''^2 + \epsilon_1\left(f'f''^2 - ff''f''' - xf''f'''\frac{\partial f}{\partial x}\right) + 2\phi\phi_1 f''^4\right\}\right] = 0, \tag{53}$$

$$\theta = 1 \quad \text{at} \quad \eta = 0,$$
$$\theta \to 0 \quad \text{as} \quad \eta \to \infty, \tag{54}$$

where $Pr = \mu c_p/k$ and $E = B^2 l^2/c_p A$ are the Prandtl number and Eckert number, respectively.

3.1.1. HAM solution

Here we take the initial guess approximation of $\theta(\eta)$ as

$$\theta_0(\eta) = e^{-\eta} \tag{55}$$

and the corresponding auxiliary linear operator

$$\mathscr{L}_2(f) = f'' - f \tag{56}$$

satisfying

$$\mathscr{L}_2[C_4 e^{\eta} + C_5 e^{-\eta}] = 0, \tag{57}$$

where C_4 and C_5 are arbitrary constants.

The zeroth-order problem is of the following form

$$(1-p)\mathscr{L}_2[\hat{\theta}(x,\eta,p) - \theta_0(\eta)] = p\hbar_2 \mathscr{N}_2[\hat{\theta}(x,\eta,p), \hat{f}(x,\eta,p)], \tag{58}$$

$$\hat{\theta}(x,0,p) = 0, \quad \hat{\theta}(x,\infty,p) = 0, \tag{59}$$

where the non-linear differential operator $\mathscr{N}_2$ is given by

$$\begin{aligned}\mathscr{N}_2[\hat{\theta}(x,\eta,p), \hat{f}(x,\eta,p)] &= \frac{\partial^2\hat{\theta}(x,\eta,p)}{\partial\eta^2} + Pr\left[\hat{f}(x,\eta,p)\frac{\partial\hat{\theta}(x,\eta,p)}{\partial\eta} - 2\frac{\partial\hat{f}(x,\eta,p)}{\partial\eta}\hat{\theta}(x,\eta,p)\right.\\ &\quad \left. + x\frac{\partial\hat{f}(x,\eta,p)}{\partial x}\frac{\partial\hat{\theta}(x,\eta,p)}{\partial\eta} - x\hat{f}'(x,\eta,p)\frac{\partial\hat{\theta}(x,\eta,p)}{\partial x}\right]\\ &\quad + PrE\epsilon_1\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\left(\frac{\partial\hat{f}(x,\eta,p)}{\partial\eta}\frac{\partial^2\hat{f}(x,\eta,p)}{\partial\eta^2}\right.\\ &\quad \left. - \hat{f}(x,\eta,p)\frac{\partial^3\hat{f}(x,\eta,p)}{\partial\eta^3} - x\hat{f}''(x,\eta,p)\hat{f}'''(x,\eta,p)\frac{\partial\hat{f}(x,\eta,p)}{\partial x}\right)\\ &\quad + PrE\left[\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2 + 2\phi\phi_1\left(\frac{\partial^2\hat{f}(x,\eta,p)}{\partial\eta^2}\right)^4\right],\end{aligned} \tag{60}$$

where $\hbar_2$ is auxiliary nonzero parameter. For $p = 0$ and $p = 1$, we respectively have

$$\hat{\theta}(x,\eta,0) = \theta_0(\eta), \quad \hat{\theta}(x,\eta,1) = \theta(x,\eta). \tag{61}$$

Obviously as p increases from 0 to 1, $\hat{\theta}(x,\eta,p)$ varies from $\theta_0(\eta)$ to $\theta(x,\eta)$. By Taylor's theorem and Eq. (60), we have

$$\hat{\theta}(x,\eta,p) = \theta_0(\eta) + \sum_{m=1}^{\infty}\theta_m(x,\eta)p^m, \tag{62}$$

where

$$\theta_m(x,\eta) = \frac{1}{m!}\left.\frac{\partial^m\hat{\theta}(x,\eta,p)}{\partial p^m}\right|_{p=0} \tag{63}$$

and convergence of series (62) depends on $\hbar_2$. Assume that $\hbar_2$ is selected such that the series (62) is convergent at $p = 1$, then due to Eq. (61) we can write

$$\theta(x,\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty}\theta_m(x,\eta). \tag{64}$$

The mth-order deformation problem is

$$\mathscr{L}_2[\theta_m(x,\eta) - \chi_m\theta_{m-1}(x,\eta)] = \hbar_2\mathscr{R}_m^2(x,\eta), \tag{65}$$
$$\theta_m(x,0) = \theta_m(x,\infty) = 0, \tag{66}$$

where

$$\begin{aligned}\mathscr{R}_m^2(x,\eta) &= \theta''_{m-1} + Pr\sum_{k=0}^{m-1}\left[f_{m-1-k}\theta'_k - 2f'_{m-1-k}\theta_k x\frac{\partial f_{m-1-k}}{\partial x}\theta'_k\right.\\ &\quad - xf'_{m-1-k}\frac{\partial\theta_k}{\partial x} + E\left\{f''_{m-1-k}f''_k + \epsilon_1\left(f'_{m-1-k}\sum_{l=0}^{k}f''_{k-l}f''_l\right.\right.\\ &\quad \left. - f_{m-1-k}\sum_{l=0}^{k}f''_{k-l}f'''_l - x\frac{\partial f_{m-1-k}}{\partial x}\sum_{l=0}^{k}f''_{k-l}f'''_l\right)\\ &\quad \left.\left. + 2\phi\phi_1 f''_{m-1-k}\sum_{l=0}^{k}f''_{k-l}\sum_{j=0}^{l}f''_{l-j}f''_j\right\}\right].\end{aligned} \tag{67}$$

The solution of the above problem can be expressed as

$$\theta_m(x,\eta) = \sum_{n=0}^{2m+2}\sum_{q=0}^{2m+2-n}A_{m,n}^q\eta^q e^{-n\eta}, \quad m \geqslant 0. \tag{68}$$

Now substituting the expression given in Eq. (68) into Eq. (65) yields the following recurrence formulae for the coefficients $A_{m,n}^q$ of $\theta_m(x,\eta)$:

for $m \geqslant 1$, $0 \leqslant n \leqslant 2m+2$ and $0 \leqslant q \leqslant 2m+2-n$:

$$A_{m,1}^0 = \chi_m\chi_{2m}A_{m-1,1}^0 - \sum_{n=2}^{2m+2}\sum_{q=0}^{2m+2-n}\Theta_{m,n}^q v_{n,0}^q, \tag{69}$$

$$A_{m,1}^k = \chi_m\chi_{2m-k}A_{m-1,1}^k + \sum_{q=k-1}^{2m+1}\Theta_{m,1}^q v_{1,k}^q, \quad 1 \leqslant k \leqslant 2m+1, \tag{70}$$

$$A_{m,n}^{k} = \chi_m \chi_{2m+1-n-k} A_{m-1,n}^{k} + \sum_{q=k}^{2m+2-n} \Theta_{m,n}^{q} v_{n,k}^{q},$$

$$2 \leqslant n \leqslant 2m+2, \ 0 \leqslant k \leqslant 2m+2-n, \tag{71}$$

$$v_{1,k}^{q} = \frac{q!2^{q+2-k}}{k!}, \quad 0 \leqslant k \leqslant 2q+2, \ q \geqslant 0, \tag{72}$$

$$v_{n,k}^{q} = \sum_{p=0}^{q+1-k} \frac{q!}{k!(n-1)^{p+1}(n+1)^{q+1-k-p}},$$

$$0 \leqslant k \leqslant 2q+2-n, \ q \geqslant 0, \ n \geqslant 2, \tag{73}$$

$$\Theta_{m,n}^{q} = \hbar_1 \Big[\chi_{2m+2-n-q} \Big(C_{m-1,n}^{q} + PrE\lambda_{m,n}^{q} \Big) + \chi_{2m+3-n-q} \Big\{ Pr \Big(\alpha_{m,n}^{q} - 2\beta_{m,n}^{q} \Big) + PrE\epsilon_1 \Big(\Pi_{m,n}^{q} - \kappa_{m,n}^{q} \Big) \Big\} + 2PrE\phi\phi_1 \Sigma_{m,n}^{q} \Big]. \tag{74}$$

The coefficients $\alpha_{m,n}^{q}$, $\beta_{m,n}^{q}$, $\kappa_{m,n}^{q}$ and $\Sigma_{m,n}^{q}$ where $m \geqslant 1$, $0 \leqslant n \leqslant 2m+2$, $0 \leqslant q \leqslant 2m+2-n$ are defined by

$$\alpha_{m,n}^{q} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+2-j\}} B_{k,j}^{i} a_{m-1-k,n-j}^{q-i}, \tag{75}$$

$$\beta_{m,n}^{q} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+2-j\}} A_{k,j}^{i} b_{m-1-k,n-j}^{q-i}, \tag{76}$$

$$\kappa_{m,n}^{q} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \times \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \times \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} d_{l,j}^{i} c_{k-l,p-j}^{t-i} a_{m-1-k,n-p}^{q-t}, \tag{77}$$

$$\Sigma_{m,n}^{q} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{j=0}^{l} \sum_{s=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+3\}} \times \sum_{w=\max\{0,q-2m+2k+1+n-s\}}^{\min\{q,2k+3-s\}} \sum_{i=\max\{0,s-2k+2l-1\}}^{\min\{s,2l+2\}} \times \sum_{r=\max\{0,w-2k+2l-1+s-i\}}^{\min\{w,2l+2-i\}} \sum_{p=\max\{0,i-2l+2j-1\}}^{\min\{i,2j+1\}} \times \sum_{t=\max\{0,r-2l+2j-1+i-p\}}^{\min\{r,2j+1-p\}} c_{j,p}^{t} c_{l-j,i-p}^{r-t} c_{k-l,s-i}^{w-r} c_{m-1-k,n-s}^{q-w}, \tag{78}$$

where the related coefficients $B_{m,n}^{k}$ and $C_{m,n}^{k}$ are given by

$$B_{m,n}^{k} = (k+1)A_{m,n}^{k+1} - nA_{m,n}^{k}, \tag{79}$$

$$C_{m,n}^{k} = (k+1)B_{m,n}^{k+1} - nB_{m,n}^{k}. \tag{80}$$

Using the above recurrence formulae, we can calculate all coefficients $A_{m,n}^{k}$ using only the first two

$$A_{0,0}^{0} = 0, \quad A_{0,1}^{0} = 1, \tag{81}$$

given by the initial guess approximations for the function $\theta(\eta)$ in Eq. (55). The corresponding Mth-order approximation of Eqs. (53) and (54) is then given by

$$\sum_{m=0}^{M} \theta_m(x,\eta) = \sum_{n=1}^{2M+2} \mathrm{e}^{-n\eta} \left(\sum_{m=n-1}^{2M+1} \sum_{k=0}^{2m+2-n} A_{m,n}^{k} \eta^{k} \right). \tag{82}$$

We obtain in fact the following explicit, totally analytic solution of the heat transfer in the PST case

$$\theta(x,\eta) = \sum_{m=0}^{\infty} \theta_m(x,\eta) = \lim_{M\to\infty} \left[\sum_{n=1}^{2M+2} \mathrm{e}^{-n\eta} \left(\sum_{m=n-1}^{2M+1} \sum_{k=0}^{2m+2-n} A_{m,n}^{k} \eta^{k} \right) \right] \tag{83}$$

and the dimensionless temperature gradient at the wall is given through

$$\theta'(x,0) = \sum_{m=0}^{\infty} \theta'_m(x,0) = \lim_{M\to\infty} \left[\sum_{n=1}^{2M+2} \sum_{m=n-1}^{2M+1} \left(A_{m,n}^{1} - A_{m,n}^{0} \right) \right]. \tag{84}$$

The dimensionless heat transfer rate at the wall, characterized by the Nusselt number Nu, is given by

$$Nu = \frac{-k\frac{\partial T}{\partial y}\Big|_{y=0}}{k(T_w - T_\infty)} x = -Re_x^{1/2} \theta'(x,0) \tag{85}$$

and the local heat flux can be expressed as

$$q_w = -k\frac{\partial T}{\partial y}\Big|_{y=0} = -kA\left(\frac{x}{l}\right)^2 \sqrt{\frac{B}{v}} \theta'(x,0). \tag{86}$$

The expressions in Eqs. (83) and (85) are evaluated for the different values of the emerging parameters and are discussed. We will now discuss the case of the prescribed heat flux in the next subsection.

3.2. The prescribed surface heat flux (PHF case)

Here the boundary conditions are of the following form:

$$-k\frac{\partial T}{\partial y} = q_w = D\left(\frac{x}{l}\right)^2 \quad \text{at} \quad y = 0, \tag{87}$$

$$T \to T_\infty \quad \text{as} \quad y \to \infty. \tag{88}$$

Taking

$$T - T_\infty = \frac{D}{k}\left(\frac{x}{l}\right)^2 \sqrt{\frac{v}{B}} g(x,\eta), \tag{89}$$

we obtain the differential equation (53) with the following boundary conditions:

$$g'(x,\eta) = -1, \quad \text{at} \quad \eta = 0,$$
$$g(x,\eta) = 0 \quad \text{as} \quad \eta \to \infty. \tag{90}$$

In this case the Eckert number is defined as

$$E = \frac{kB^2 l^2}{Dc_p}\sqrt{\frac{B}{\nu}}. \tag{91}$$

3.2.1. HAM solution

We note that solution here is the same as in the previous subsection except that the recurrence relation in Eq. (69) now is

$$A^0_{m,1} = \chi_m \chi_{2m} A^0_{m-1,1} - \sum_{q=0}^{2m+1} \Theta^q_{m,1} v^q_{1,1} - \sum_{n=2}^{2m+2}\left[n\Theta^0_{m,n} v^0_{n,0} + \sum_{q=1}^{2m+2-n} \Theta^q_{m,n}(nv^q_{n,0} - v^q_{n,1})\right]. \tag{92}$$

The corresponding Mth-order approximation of Eqs. (53) and (90) is

$$\sum_{m=0}^{M} g_m(x,\eta) = \sum_{n=1}^{2M+2} e^{-n\eta}\left(\sum_{m=n-1}^{2M+1}\sum_{k=0}^{2m+2-n} A^k_{m,n}\eta^k\right) \tag{93}$$

and totally analytic solution of the heat transfer in the PHF case is

$$g(x,\eta) = \sum_{m=0}^{\infty} g_m(x,\eta) = \lim_{M\to\infty}\left[\sum_{n=1}^{2M+2} e^{-n\eta}\left(\sum_{m=n-1}^{2M+1}\sum_{k=0}^{2m+2-n} A^k_{m,n}\eta^k\right)\right]. \tag{94}$$

The wall temperature T_w is obtained from Eq. (89) as

$$T - T_\infty = \frac{D}{k}\left(\frac{x}{l}\right)^2\sqrt{\frac{\nu}{B}}g(x,0). \tag{95}$$

4. The convergence of the solution

The explicit, analytic expressions (46), (83) and (94) contain two auxiliary parameters $\hbar_1$ and $\hbar_2$. As pointed out by Liao [14], the convergence region and rate of approximations given by the homotopy analysis method are strongly dependent upon these auxiliary parameters. In Fig. 1(a–c) the $\hbar$-curves are plotted to see the range of admissible values for the parameters $\hbar_1$ and $\hbar_2$. It is clear from Fig. 1(a–c) that the range for the admissible values for $\hbar_1$ and $\hbar_2$ is $-1 \leqslant \hbar_1, \hbar_2 < 0$. And the series given by Eqs. (46), (83) and (94) converges in the whole region of η, when

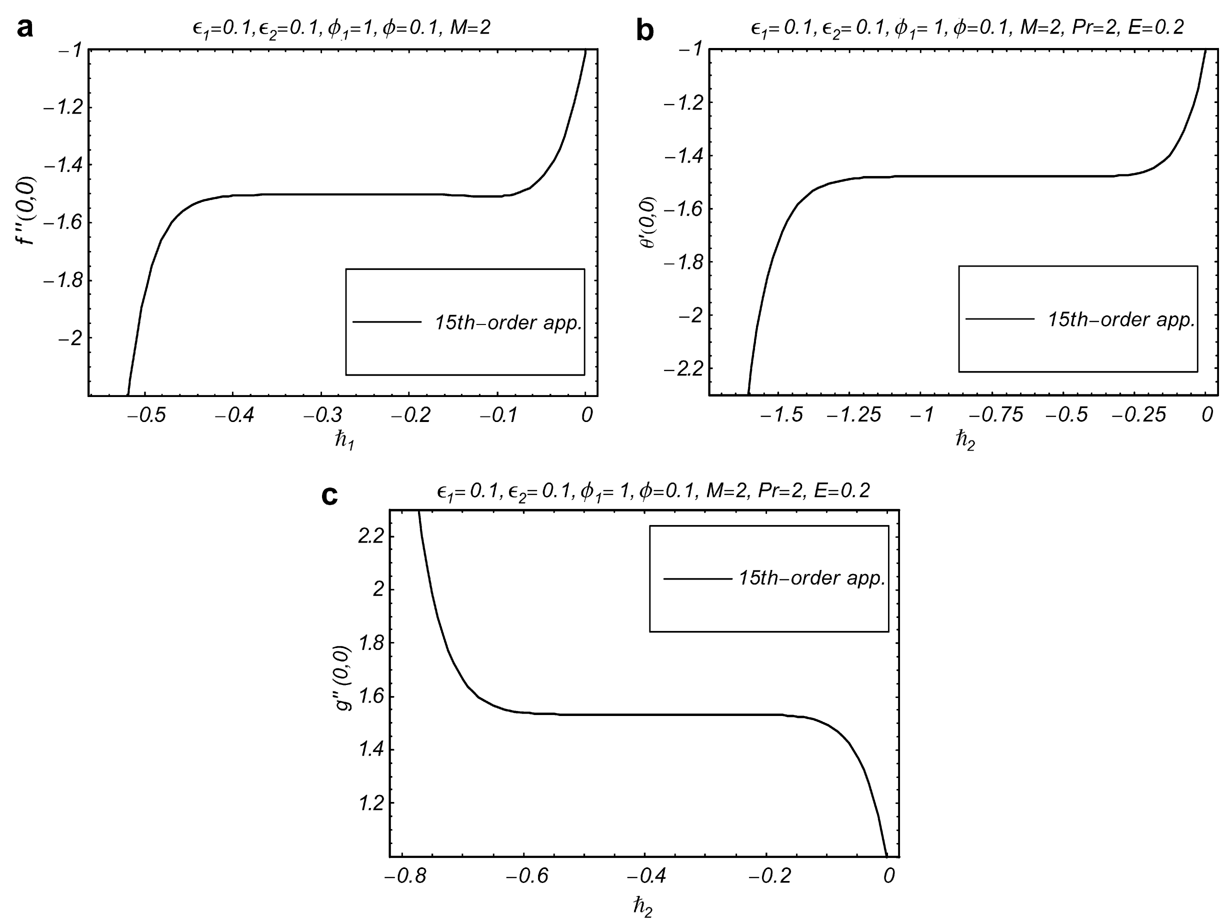

Fig. 1. $\hbar$-Curves are plotted for the functions f, θ and g: (a) flow analysis, (b) PST case and (c) PHF case.

$\hbar_1 = -0.4$, $\hbar_2 = -0.75$ for PST case and $\hbar_2 = -0.4$ for PHF case. It is also observed that the series (46) of $f(x,\eta)$ converges faster than that of the $\theta(x,\eta)$ and $g(x,\eta)$. This is due to the fact that the non-linearity in the later case is stronger than the former.

5. Results and discussion

In this section, attention has been focused to the variations of ϕ, ϕ_1, M, Pr, E, ϵ_1 and ϵ_2. For this purpose Figs. 2–7 have been displayed. In order to see the variation of

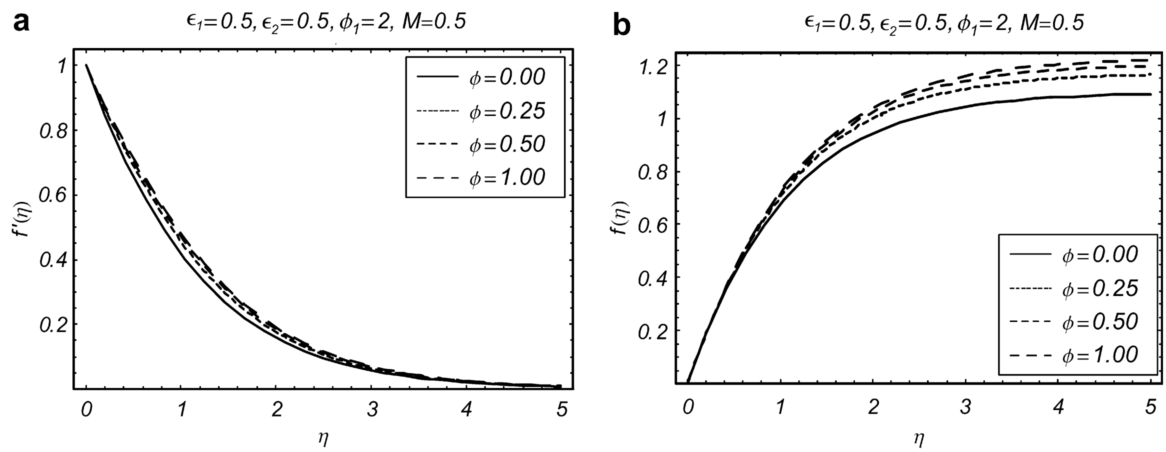

Fig. 2. Variation of the dimensionless velocity fields f' and f with increasing third-order parameter ϕ: (a) $f'(\eta)$ and (b) $f(\eta)$.

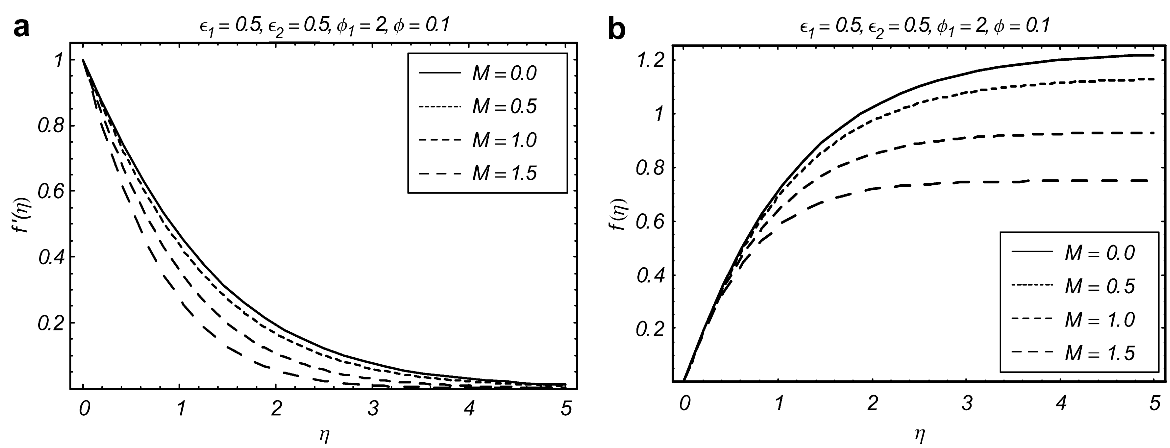

Fig. 3. Variation of the dimensionless velocity fields f' and f with increasing MHD parameter M: (a) $f'(\eta)$ and (b) $f(\eta)$.

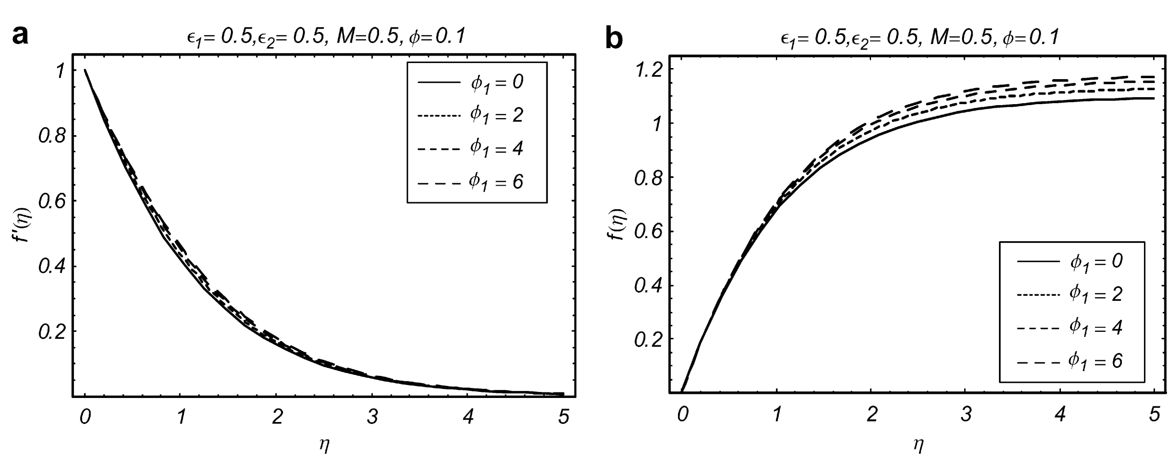

Fig. 4. Variation of the dimensionless velocity fields f' and f with increasing parameter ϕ_1: (a) $f'(\eta)$ and (b) $f(\eta)$.

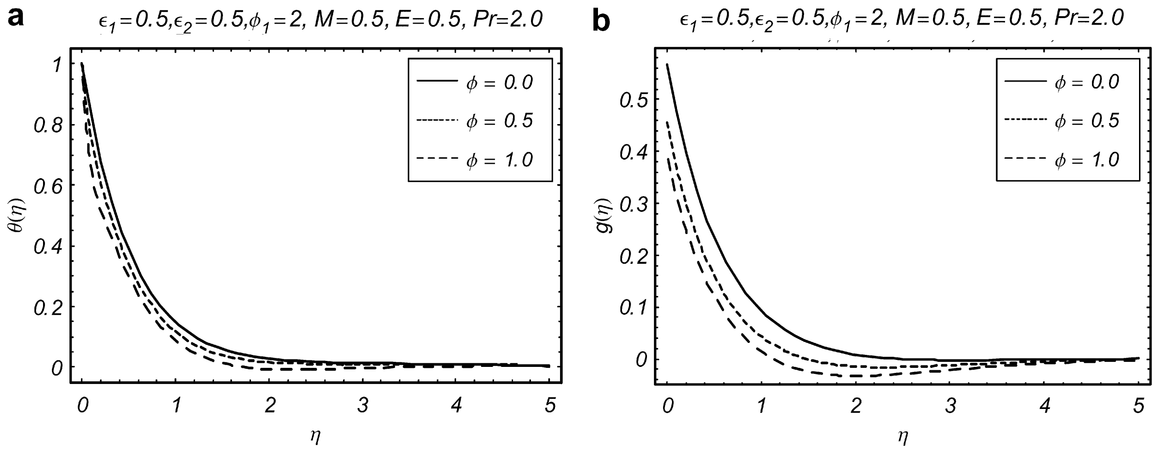

Fig. 5. Variation of the dimensionless temperature profiles θ and g with increasing third-order parameter ϕ: (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

third-order parameter ϕ, ϕ_1 and magnetic parameter M on velocity components u and v, the respective graphs for $f'(x,\eta)$ and $f(x,\eta)$ have been sketched in Figs. 2–4. The graphs for the variation of ϕ, ϕ_1, M, Pr and E on the temperature are shown in Figs. 5–9. In these figures, $\theta(x,\eta)$ is the temperature variation that corresponds to the PST case and $g(\eta)$ is the temperature variation for the PHF case. Moreover, the variations of ϕ and M on the skin friction

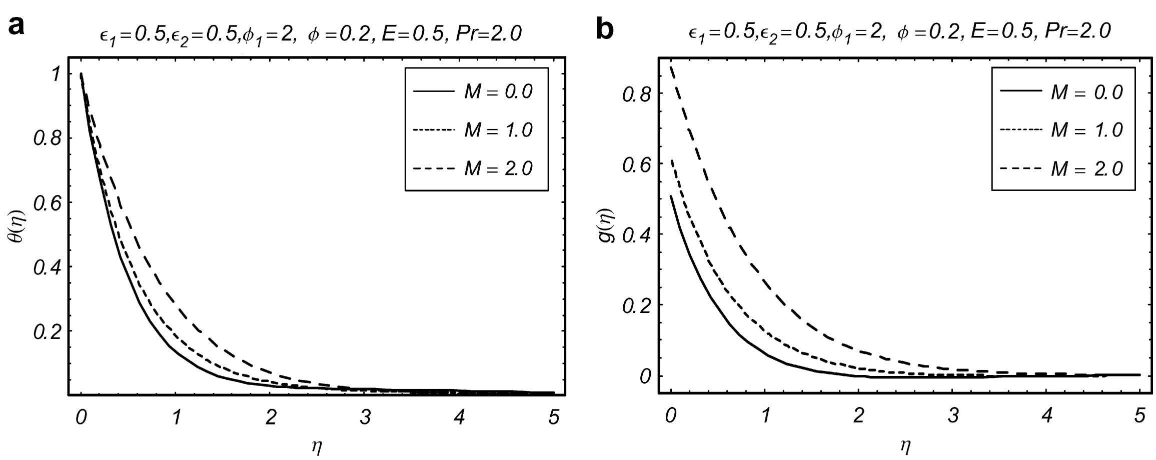

Fig. 6. Variation of the dimensionless temperature profiles θ and g with increasing MHD parameter M: (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

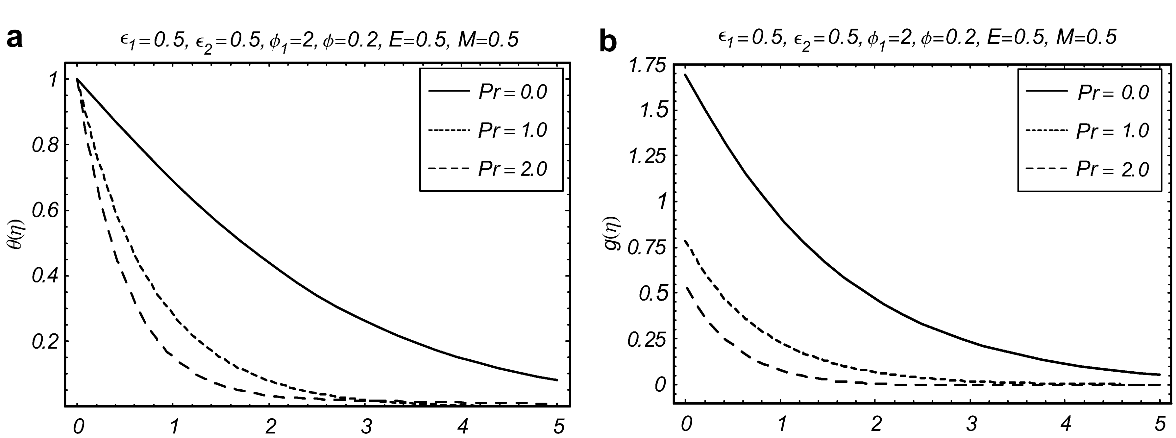

Fig. 7. Variation of the dimensionless temperature profiles θ and g with increasing Prandtl number Pr: (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

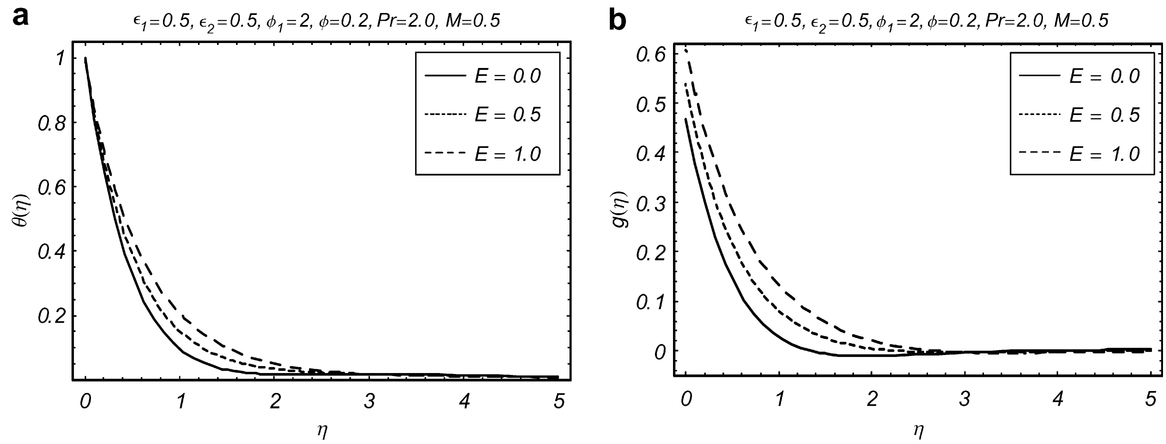

Fig. 8. Variation of the dimensionless temperature profiles θ and g with increasing Eckert number E: (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

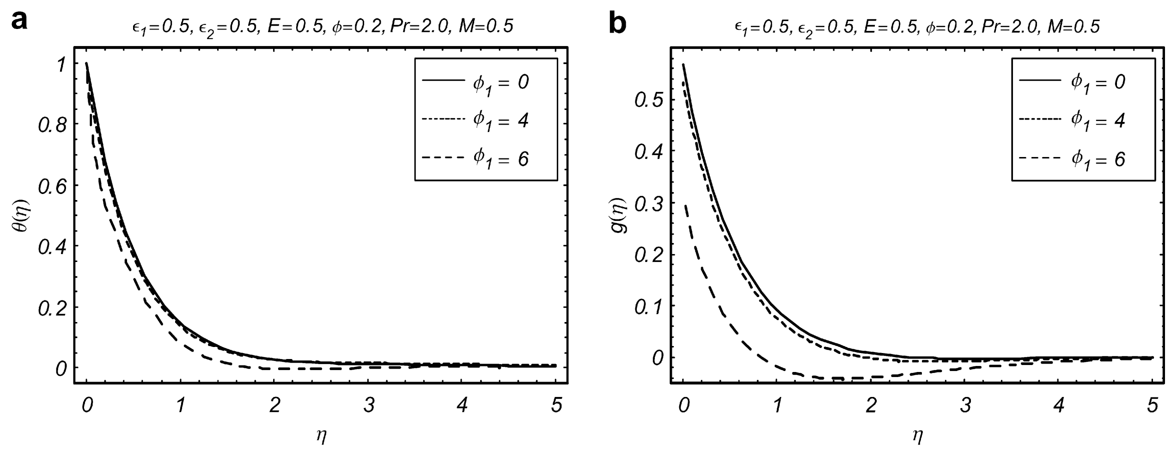

Fig. 9. Variation of the dimensionless temperature profiles θ and g with increasing parameter ϕ_1: (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

Table 1
Values of the skin friction coefficient $C_f Re_x^{1/2}$ for $\epsilon_1 = 0.2$, $\epsilon_2 = 0.2$ and $\hbar_1 = -0.3$

ϕ	$M = 0.0$	$M = 0.5$	$M = 0.75$	$M = 1.0$
0.00	−2.26495	−2.51797	−2.80760	−3.17236
0.50	−2.32154	−2.58969	−2.90186	−3.30240
0.15	−2.35527	−2.62746	−2.94653	−3.35940
0.20	−2.41114	−2.68147	−3.00023	−3.41655
0.25	−2.51679	−2.77597	−3.08254	−3.48638

Table 2
Values of the skin friction coefficient $C_f Re_x^{1/2}$ for $\epsilon_2 = 0.2$, $\phi = 0.0$ and $\hbar_1 = -0.3$

ϵ_1	$M = 0.0$	$M = 0.5$	$M = 0.75$	$M = 1.0$
0.0	−1.79130	−1.98445	−2.20682	−2.48859
0.1	−2.04611	−2.27220	−2.53167	−2.85931
0.2	−2.26495	−2.51797	−2.80760	−3.17236
0.3	−2.46548	−2.74155	−3.05697	−3.45352
0.4	−2.65330	−2.95004	−3.28868	−3.71404

Table 3
Values of the skin friction coefficient $C_f Re_x^{1/2}$ for $\epsilon_1 = 0.2$, $\phi = 0.0$ and $\hbar_1 = -0.3$

ϵ_2	$M = 0.0$	$M = 0.5$	$M = 0.75$	$M = 1.0$
0.0	−2.45711	−2.72905	−3.04098	−3.43464
0.1	−2.35718	−2.61941	−2.91989	−3.29869
0.2	−2.26495	−2.51797	−2.80760	−3.17236
0.3	−2.17988	−2.42417	−2.70355	−3.05507
0.4	−2.10144	−2.33748	−2.60718	−2.94623

Table 4
Values of the Nusselt number $-Re_x^{-1/2} Nu$ for $\epsilon_1 = 0.2$, $\epsilon_2 = 0.2$, $\phi = 0.1$, $M = 0.5$, $\hbar_2 = -0.75$ and $\hbar_1 = -0.3$

Pr	$E = 0.0$	$E = 0.2$	$E = 0.4$	$E = 0.6$
0.5	0.92382	0.89027	0.85671	0.82316
1.0	1.39525	1.33638	1.27752	1.21865
1.5	1.76940	1.69097	1.61254	1.53412
2.0	2.07285	1.97850	1.88414	1.78978

coefficient have been listed in Table 1. Tables 2 and 3 have been prepared to show the variation of ϵ_1, ϵ_2 and M on the skin friction coefficient. Table 4 presents the variations of Pr and E on the Nusselt number. From the present study, the main findings can be summarized as follows:

- Increasing ϕ, the x-component of velocity and boundary layer thickness increases.
- The y-component of velocity increases and boundary layer thickness decreases for large values of ϕ.
- The x-component of velocity decreases and boundary layer thickness increases when value of M is increased.
- Large values of M decrease the y-component of velocity and the boundary layer thickness.
- An increase in the value of ϕ leads to a decrease of the temperature. But the thermal boundary layer thickness increases by increasing ϕ.
- The behaviour of M on the temperature and the thermal boundary layer thickness is quite opposite to that of ϕ.
- The variation of Pr for the temperature and thermal boundary layer thickness is similar to that of ϕ.
- The influence of E on the temperature and thermal boundary layer thickness is similar to that of M but is quite opposite of Pr.
- The effect of the parameter ϕ_1 on the velocity and temperature are similar to that of the third-order parameter ϕ.
- The magnitude of skin friction coefficient increases by increasing ϕ and ϵ_1 keeping M fixed.
- When ϵ_2 increases, the magnitude of skin friction coefficient decreases by keeping M fixed.
- For fixed M, the variation of ϵ_2 on the magnitude of skin friction coefficient is quite opposite of ϵ_1.
- The variation of M on the magnitude of skin friction is similar to that of ϕ.
- The Nusselt number decreases by increasing E and fixed Pr and the effect is opposite for increasing Pr and fixed E.
- The HAM solutions for the second grade and Newtonian fluids with heat transfer can be taken as the limiting cases by choosing $\phi = 0$ and $\phi = \epsilon_1 = \epsilon_2 = 0$, respectively.

6. Concluding remarks

In this paper, we have considered a problem concerning the MHD flow and heat transfer analysis of the third-order fluid. The solution for boundary layer flow caused by a stretching sheet is obtained. To carry out heat transfer analysis, the energy equation has been solved for the prescribed surface temperature and heat flux cases. Analytical solutions for the velocity and temperature distributions are obtained using an analytical technique, namely the homotopy analysis method [14,15]. The convergence of the results are shown. The results are presented graphically and the effects of the emerging parameters are seen. The skin friction coefficient and the Nusselt number are tabulated. It is interesting to note that the velocity increases for the large value of the third-order parameter ϕ. Also, the temperature decreases with an increase in the third-order parameter. To the best of our knowledge, such analytic solutions have never been reported. The obtained solutions have promising applications in engineering such as materials manufactured by extrusion process, on conveyer belts etc. Such results should be applicable for a variety of non-Newtonian fluids such as aqueous solutions of high molecular weight, polyethylene oxide and polyacrylamide.

References

[1] W.C. Tan, T. Masuoka, Stokes first problem for second grade fluid in a porous half space, Int. J. Non-Linear Mech. 40 (2005) 515–522.
[2] C. Fetecau, C. Fetecau, Starting solutions for some unsteady unidirectional flows of a second grade fluid, Int. J. Eng. Sci. 43 (2005) 781–789.
[3] C. Fetecau, J. Zierep, On a class of exact solutions of the equations of motion of a second grade fluid, Acta Mech. 150 (2001) 135–138.
[4] C. Fetecau, C. Fetecau, J. Zierep, Decay of a potential vortex and propagation of a heat wave in a second grade fluid, Int. J. Non-Linear Mech. 37 (2002) 1051–1056.
[5] T. Hayat, S. Asghar, A.M. Siddiqui, Some non-steady flows of a non-Newtonian fluid, Int. J. Eng. Sci. 38 (2000) 337–346.
[6] T. Hayat, Y. Wang, K. Hutter, Hall effects on the unsteady hydromagnetic oscillatory flow of a second grade fluid, Int. J. Non-Linear Mech. 39 (2004) 1027–1037.
[7] W.C. Tan, T. Masuoka, Stokes first problem for an Oldroyd-B fluid in a porous half space, Phys. Fluid. 17 (2005) 023101–023107.
[8] T. Hayat, Oscillatory solution in rotating flow of a Johnson–Segalman fluid, Z. Angew. Math. Mech. 85 (2005) 449–456.
[9] I. Chung Liu, Flow and heat transfer of an electrically conducting fluid of second grade in a porous medium over a stretching sheet subject to a transverse magnetic field, Int. J. Non-Linear Mech. 40 (2005) 465–474.
[10] G.S. Beavers, D.D. Joseph, The rotating rod viscometer, J. Fluid Mech. 69 (1975) 475–511.
[11] D.D. Joseph, G.S. Beavers, R.I. Fosdick, The free surface on a liquid between cylinders rotating at different speeds, Arch. Rat. Mech. Anal. 49 (1973) 321–401.
[12] C. Bruce, Ph.D. Dissertation Stanford University, Stanford, CA, 1969.
[13] D.D. Joseph, Fluid Dynamics of Viscoelastic Fluids, Springer, New York, Berlin, Heidelberg, 1980.
[14] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Dissertation (in English), Shanghai Jiao Tong University, Shanghai, 1992.
[15] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.
[16] S.J. Liao, Application of process analysis method to the solution of 2D nonlinear progressive gravity waves, J. Ship. Res. 36 (1992) 30–37.
[17] S.J. Liao, A second-order approximate analytical solution of a simple pendulum by the process analysis method, J. Appl. Mech. 59 (1992) 970–975.
[18] S.J. Liao, An approximate solution technique which does not depend upon small parameters: a special example, Int. J. Non-Linear Mech. 30 (1995) 371–380.
[19] S.J. Liao, A kind of approximate solution technique which does not depend upon small parameters (part 2): an application in fluid mechanics, Int. J. Non-Linear Mech. 32 (1997) 815–822.
[20] S.J. Liao, A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.

[21] S.J. Liao, An explicit, totally analytic approximate solution for Blasius' viscous flow problems, Int. J. Non-Linear Mech. 34 (1999) 759–778.
[22] S.J. Liao, On a generalized Taylor theorem: a rational proof of the validity of the so-called homotopy analysis method, J. Appl. Math. Mech. 24 (2003) 47–54.
[23] S.J. Liao, A new branch of solutions of boundary-layer flows over an impermeable stretched plate, Int. J. Heat Mass Transfer 48 (2005) 2529–2539.
[24] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.
[25] S.J. Liao, An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate, Commun. Non-linear Sci. Numer. Simm. 11 (2006) 326–339.
[26] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Int. J. Eng. Sci. 41 (2003) 2091–2103.
[27] T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Int. J. Eng. Sci. 42 (2004) 123–135.
[28] T. Hayat, M. Khan, M. Ayub, Couette and Poiseuille flows of an Oldroyd 6-constant fluid with magnetic field, J. Math. Anal. Appl. 298 (2004) 225–244.
[29] T. Hayat, M. Khan, S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, Acta Mech. 168 (2004) 213–232.
[30] H. Xu, An explicit analytic solution for convective heat transfer in an electrically conducting fluid at a stretching surface with uniform free stream, Int. J. Eng. Sci. 43 (2005) 859–874.
[31] M. Sajid, T. Hayat, S. Asghar, On the analytic solution of the steady flow of a fourth grade fluid, Phys. Lett. A 355 (2006) 18–24.
[32] M. Pakdemirli, The boundary layer equations of third grade fluids, Int. J. Non-Linear Mech. 27 (1992) 785–793.